The Two Loop Crossed Ladder Vertex Diagram with Two Massive Exchanges

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Abstract

We compute the (three) master integrals for the crossed ladder diagram with two exchanged quanta of equal mass. The differential equations obeyed by the master integrals are used to generate power series expansions centered around all the singular (plus some regular) points, which are then matched numerically with high accuracy. The expansions allow a fast and precise numerical calculation of the three master integrals (better than 15 digits with less than 30 terms in the whole real axis). A conspicuous relation with the equal-mass sunrise in two dimensions is found. Comparison with a previous large momentum expansion is made finding complete agreement.

Key words: Feynman diagrams, Multi-loop calculations

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1 Introduction

Electron-positron linear colliders of next generation with a large energy, $E = \mathcal{O}\left(1\,\text{TeV}\right)$, and a large luminosity, $L = \mathcal{O}\left(10^{34}\text{cm}^{-2}\text{s}^{-1}\right)$, will produce an incredible amount of accurate data [1]. Theoretical work in the same direction, i.e. pointing to accurate computations of electroweak cross sections and decay widths, is therefore mandatory. That means, in practice, to go beyond the standard one-loop approximation and address the control of the two-loop quantum corrections. While in QCD multi-loop computations are often feasible by setting to zero the small quark masses simplifying then the problem to a massless one, in the electroweak case setting to zero the masses of the W, Z, Higgs or top quark is not possible. One has to face the general problem of computing massive multiscale Feynman diagrams. In many cases, the full computation can be drastically simplified by finding a small expansion parameter η, such as the ratio of particle masses and/or kinematical invariants:

$$\eta = \frac{m^2}{M^2} \ll 1\,. \tag{1}$$

The heavy algebra entering the higher orders in η can be dealt with using standard algebraic programs such as Form [2] or Mathematica [3]. In problems involving many different scales or, equivalently, many different dimensionless ratios, that is probably the only viable analytical way. Another strategy is that of numerical computation of the integrals "from the beginning", such as for example the numerical computation of loop integrals written in terms of Feynman parameters [4]. This approach is not easy to implement when a subset of particles is massless, because of troublesome infrared effects. A different strategy assumes the dominance of the logarithmic terms over constants and power-suppressed terms in the high-energy cross-sections, i.e. of terms of the form [5]

$$\left(\frac{g^2}{16\pi^2}\right)^n \log^k \frac{s}{m^2}, \qquad (k = 1, 2, \cdots 2n), \tag{2}$$

where g is the weak coupling and m is the W or Z^0 mass. The logarithmic terms are computed directly by means of asymptotic expansions [6, 7]. This strategy is similar in spirit to the one used in QCD to construct for example shower Montecarlo programs [8]. The logarithmic terms are certainly the dominant ones in the formal limit $s \to \infty$ (i.e. $s \gg m$), but at a fixed energy non logarithmic terms may in general have a significant numerical effect. In cases in which one or (at most) two dimensionless quantities of order one are involved, an exact analytic computation may be useful. A model computation for higher-order electroweak effects is the two-loop form factor in the degenerate mass limit,

$$m_W = m_Z = m_H = m. \tag{3}$$

In [9, 10] two of us have computed the master integrals of the two-loop electroweak form factor for all diagrams with the exception of the crossed-ladder with the exchange of two equal mass quanta — such as a pair of W's and/or Z's (see fig. 1). These master integrals are naturally expressed in the basis of the harmonic polilogarithms (HPLs), transcendental functions verifying algebra properties which largely simplify the transformations [11]. Some master integrals involving two or three massive propagators have required a generalization

of the harmonic polilogarithms basis, involving basic functions with square roots — the so-called generalized harmonic polilogarithms (GHPLs) [10].

In this paper we compute the last missing piece in order to obtain an exact analytic expression of the two-loop electro-weak form factor: the crossed ladder diagram with two equal-mass quanta exchanged. By standard reduction techniques, we find that this topology involves three master integrals. At variance with respect to the planar topologies and the non-planar topology with at most one massive exchange, it is not possible to represent any of these master integrals in terms of harmonic polylogarithms or generalized harmonic polylogarithms: elliptic integrals are involved, i.e. integrals of square roots of polynomials of degree three or four [12]. For that reason, we use a different method, already applied in [13, 14] to evaluate the equal-mass sunrise diagram. The differential equations obeyed by the master integrals are used to generate power-series expansions centered around all the singular points of the differential equations themselves. It may also be convenient to generate auxiliary expansions around regular points, which are in principle completely arbitrary. The series expansions centered around two different points are then matched numerically in a point belonging to the intersection of the respective domains of convergence. In general, we are not able to find a closed analytic expression for the master integrals, but only truncated series representations. The latter allow, however, a fast and accurate numerical evaluation of the master integrals themselves (better than 15 digits with less than 30 terms, typically, for the "accelerated" versions of the expansions). The logarithmic terms in the large momentum expansion of one of the master integrals (the basic scalar amplitude) have been obtained in [15]. Our expansion is in agreement with these results; we are also able to compute the power-suppressed corrections.

2   Power Series Solution of the Differential Equations on the MIs

In this section we outline the computation of the master integrals of the two-equal-mass-exchanged crossed-ladder diagram, which is described extensively in the next sections. As anticipated in the introduction, we compute the master integrals with a general semi-analytical method [12], which can be applied to arbitrarily complicated cases. The first steps (sec. 4) are common to the method used to compute the previous electroweak form factor diagrams [9, 10]. By using standard reduction procedures, the problem is shifted to that of computing three independent scalar amplitudes, called master integrals (MIs), which obey a system of coupled first-order differential equations in the evolution variable $x \equiv -s/m^2$, where $s = E_{cm}^2$ is the Mandelstam variable (sec. 5). These equations are linear and not homogeneous. By a suitable choice of the master integral basis, we succeeded in triangularizing the system to a system of two coupled equations (let's say the first two equations on F_1 and F_2) plus a decoupled equation from the previous ones (let's say the third one on F_3):

$$\frac{dF_1}{dx} = A_{11}(x)F_1(x) + A_{12}(x)F_2(x) + \Omega_1(x); \tag{4}$$

$$\frac{dF_2}{dx} = A_{21}(x)F_1(x) + A_{22}(x)F_2(x) + \Omega_2(x); \tag{5}$$

$$\frac{dF_3}{dx} = A_{31}(x)F_1(x) + A_{32}(x)F_2(x) + A_{33}(x)F_3(x) + \Omega_3(x), \tag{6}$$

where the $A_{ij}(x)$'s, the coefficients of the associated homogeneous system, are rational fractions in x, while the $\Omega_k(x)$'s are known functions (logarithms, polylogarithms etc.). We then derive from the system of the two coupled first-order equations a single second-order equation in one of the two master integrals involved, let's say F_1 (sec. 6):

$$\frac{d^2 F_1}{dx^2} + A(x)\frac{dF_1}{dx} + B(x)F_1(x) = \Omega(x). \tag{7}$$

The resulting equation belongs to the Fuchs class, i.e. it has *regular singular* points only, including also the point at infinity[1] [12]. There are *four* such singularities, all located on the real axis, at:

$$x = 0; \quad x = 8; \quad x = -1; \quad x = \infty. \tag{8}$$

The differential equation (7) is used to generate power series expansions centered around all these singular points (sec. 7). Auxiliary expansions around (arbitrary) regular points may also be of utility. Each series is determined by the differential equation up to two arbitrary coefficients, which must be fixed by some initial or boundary condition. A qualitative knowledge of the behaviour of the master integral $F_1(x)$ around $x = 0$ is sufficient to fix exactly (in fact analytically) the two arbitrary coefficients of the expansion around this point. The coefficients of the series centered around $x = 0$ (small-momentum expansion) are therefore completely determined in analytical way. The arbitrary coefficients occurring in the serieses centered in the remaining points are found by imposing matching conditions in suitable points belonging to the intersections of the respective domains of convergence. Assume that one knows the expansions of $F_1(x)$ around two different points x_1 and x_2

$$F_1(x) = \sum_{n=0}^{\infty} a_n^{(1)}(x - x_1)^n \quad \text{for} \quad |x - x_1| < R_1, \tag{9}$$

$$F_1(x) = \sum_{n=0}^{\infty} a_n^{(2)}(x - x_2)^n \quad \text{for} \quad |x - x_2| < R_2, \tag{10}$$

with $|x_1 - x_2| < R_1 + R_2$, so that that the two circles overlap, and that the coefficients $a_n^{(1)}$ of the first series are completely known, while those of the second series are determined as (linear) functions of the first two by the underlying differential equation:

$$a_n^{(2)} = b_n\, a_0^{(2)} + c_n\, a_1^{(2)} + d_n. \tag{11}$$

---

[1] In general, by a singularity of the differential equation we mean a singularity in the coefficients $A(x)$ or $B(x)$. A point x_0 is a *regular singular point* of a differential equation if its solutions can be written as singular factor $\sim (x - x_0)^\alpha$ or $\sim (x - x_0)^\alpha \log(x - x_0)$ multiplied by convergent serieses in a neighborhood of x_0.

By taking a point $\bar{x}$ in the intersection domain, the matching conditions give a linear system on the unknowns $a_0^{(2)}$ and $a_1^{(2)}$.

$$
a_0^{(2)} \sum_{n=0}^{\infty} b_n \left(\bar{x} - x_2\right)^n + a_1^{(2)} \sum_{n=0}^{\infty} c_n \left(\bar{x} - x_2\right)^n =
$$
$$
\sum_{n=0}^{\infty} a_n^{(1)} \left(\bar{x} - x_1\right)^n - \sum_{n=0}^{\infty} d_n \left(\bar{x} - x_2\right)^n ; \tag{12}
$$
$$
a_0^{(2)} \sum_{n=1}^{\infty} n \, b_n \left(\bar{x} - x_2\right)^{n-1} + a_1^{(2)} \sum_{n=1}^{\infty} n \, c_n \left(\bar{x} - x_2\right)^{n-1} =
$$
$$
\sum_{n=1}^{\infty} n \, a_n^{(1)} \left(\bar{x} - x_1\right)^{n-1} - \sum_{n=1}^{\infty} n \, d_n \left(\bar{x} - x_2\right)^{n-1} . \tag{13}
$$

Once $a_0^{(2)}$ and $a_1^{(2)}$ are known, one determines the higher-order coefficients $a_n^{(2)}$ for $n > 1$. Instead of matching the derivatives of the serieses, one can also take a second matching point $\bar{x}'$.

Once the first master integral F_1 has been determined, one plugs its expression into the first-order equation (4) of the 2×2 system involving dF_1/dx and determines algebraically F_2 (sec. 8). One finally inserts the expression of the first two MIs F_1 and F_2 into the third differential equation (6) and determines F_3 by quadrature. Let us remark that one can study all the analytical properties of the MIs (analytic continuation, behaviour close to thresholds, asymptotic properties, etc.) by looking at their power-series expansions. The latter also offer a suitable and powerful way for the precise and fast numerical evaluation of the MIs, for example with a Fortran routine.

In sec. 9 we discuss the relation between the differential equation on the master integral F_1 for the crossed ladder and that one for the sunrise diagram with three equal masses in two space-time dimensions [13]. The "physical" origin of this connection is, at present, unclear. Finally, in section 10 we draw our conclusions.

3   Threshold Structure

The crossed ladder diagram with two equal-mass quanta exchanged has a rather simple threshold structure: there is a 2-particle cut on the internal massless lines for

$$
s \geq 0 \tag{14}
$$

and two 3-particle cuts on the internal massless lines and one of the massive lines for

$$
s \geq m^2. \tag{15}
$$

There is also a pseudothreshold for $s \leq 4m^2$. The complexity of the differential equations and of the structure of the master integral cannot be guessed by looking at the threshold structure of the diagram. That is to be contrasted to the case of the equal-mass sunrise, where the threshold at $s = 9m^2$ as well as the pseudothreshold in $s = m^2$ are identifiable as the basic sources of complexity.
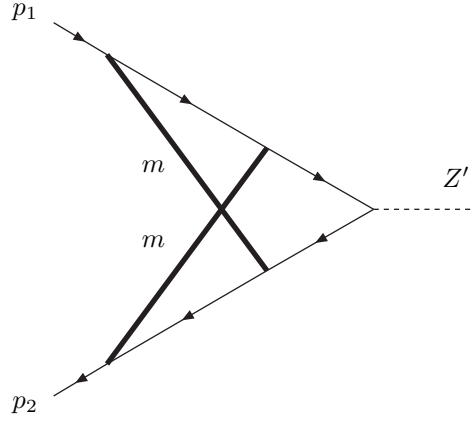
Figure 1: *Feynman diagram for the annihilation of a pair of massless fermions with the exchange of two massive quanta with equal mass m. The thin lines represent the massless fermions, while the thick lines represent the massive quanta. The outgoing dashed line represents the probe (for instance a Z').*

4   Reduction to Master Integrals

By standard decomposition into invariant form factors and rotation of the scalar products, one can show that the computation of the two equal-mass crossed ladder diagram (see fig. 1) is equivalent to the computation of the following independent scalar amplitudes:

$$F\left(n_1, n_2, n_3, n_4, n_5, n_6, s\right) = \int \frac{S^r}{P_1^{n_1} \, P_2^{n_2} \, P_3^{n_3} \, P_4^{n_4} \, P_5^{n_5} \, P_6^{n_6}} \mathcal{D}^D k_1 \, \mathcal{D}^D k_2, \qquad (16)$$

where D is the space-time dimension, the scalar product is defined as

$$a \cdot b \equiv \vec{a} \cdot \vec{b} - a_0 \, b_0, \qquad (17)$$

the loop measure is

$$\mathcal{D}^D k \equiv \frac{1}{\Gamma(3 - D/2)} \frac{d^D k}{4\pi^{D/2}}, \qquad (18)$$

with $\Gamma(z)$ the Euler Gamma function. We consider a routing of the loop momenta k_1^μ and k_2^ν which results in the following denominators:

$$
\begin{aligned}
P_1 &= k_1^2 + m^2, & (19)\\
P_2 &= k_2^2 + m^2, & (20)\\
P_3 &= (p_1 - k_1)^2, & (21)\\
P_4 &= (p_2 - k_2)^2, & (22)\\
P_5 &= (p_1 - k_1 + k_2)^2, & (23)\\
P_6 &= (p_2 + k_1 - k_2)^2, & (24)
\end{aligned}
$$

and the following irreducible numerator (scalar product):

$$S = p_2 \cdot k_1. \qquad (25)$$

The indices of the denominators are assumed to be all positive[2], $n_i > 0$ while the index of the scalar product can be positive or zero, $r \geq 0$.

---

[2] If $n_i \leq 0$ for some i we have a sub-topology in which line i is shrinked to a point.

The above independent scalar amplitudes, considered as functions of the integer indices n_i's and r, are related to each other by integral identities obtained in general by means of integration by parts over the loop momenta, invariance under Lorentz transformations and symmetry relations coming from the particular mass distribution of the diagram under consideration [16]. These identities can be used to express a given amplitude F as a linear combination of a set of reference amplitudes, with fixed indices, called master integrals (MI) F_i:

$$F\left(n_1, n_2, n_3, n_4, n_5, n_6, r\right) = \sum_{i=1}^{N} c_i\left(n_1, n_2, n_3, n_4, n_5, n_6, r\right) F_i, \qquad (26)$$

where the c_i's are known coefficients (rational fractions in x and D). The above reduction involves the solution of the integration-by-parts identities in some recursive way over the indices. Nowadays the most common algorithm is the "Laporta algorithm" [17], which we have used for the crossed ladder topology. That way we have been able to reduce the independent amplitudes to a linear combination of three master integrals:

$$F\left(n_1, n_2, n_3, n_4, n_5, n_6, s\right) = \sum_{i=1}^{3} c_i\left(n_1, n_2, n_3, n_4, n_5, n_6, s\right) F_i + \cdots, \qquad (27)$$

where the dots denote contributions from master integrals with less than six denominators, i.e. from subtopologies, which are known [9, 10, 18]. Let us choose the following basis of master integrals:

$$F_1(x; \epsilon) = \int \frac{1}{P_1 P_2 P_3 P_4 P_5 P_6} \mathcal{D}^D k_1 \mathcal{D}^D k_2; \qquad (28)$$

$$F_2(x; \epsilon) = \int \frac{1}{P_1^2 P_2 P_3 P_4 P_5 P_6} \mathcal{D}^D k_1 \mathcal{D}^D k_2; \qquad (29)$$

$$F_3(x; \epsilon) = \int \frac{S}{P_1 P_2 P_3 P_4 P_5 P_6} \mathcal{D}^D k_1 \mathcal{D}^D k_2. \qquad (30)$$

The first MI is the basic scalar amplitude, with linear denominators and no scalar products; the second MI contains the first denominator squared, while the third MI involves an irreducible numerator. By direct inspection, one finds that the above MI's do not contain any pole in $D - 4$ because they are ultraviolet finite and, since the radiated quanta are massive, there is an effective infrared cut-off $\approx m$ which renders the MIs infrared finite as well. Let us therefore set $D = 4$ from now on and omit this argument.

5 The System of Differential Equations

Once the reduction to master integrals has been achieved, the following step is the actual computation of the master integrals themselves. We use the differential equation method [19], which involves taking a derivative with respect to

$$x = -\frac{s}{m^2}, \qquad (31)$$

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at fixed m^2, of the master integral F_i:

$$x\,\frac{\partial F_i}{\partial x} \;=\; \frac{1}{2}\,p_1^\mu\,\frac{\partial F_i}{\partial p_1^\mu} \;=\; \frac{1}{2}\,p_2^\mu\,\frac{\partial F_i}{\partial p_2^\mu}\,. \tag{32}$$

$q = p_1 + p_2$ the momentum of the probe and $q^2 = -s$, s being the squared c.m. energy. By taking the derivative inside the integral in eq. (32), various amplitudes are generated, which are reduced again to master integrals as discussed in the previous section. We then obtain in general a system of linear differential equations with variable coefficients, of the form:

$$\frac{d}{dx}F_i(x) \;=\; \sum_{j=1}^{3} f_{ij}(x)\,F_j(x) \;+\; N_i(x) \qquad (i = 1,2,3)\,. \tag{33}$$

By changing the basis for the master integrals, the functions $f_{ij}(x)$ and $N_i(x)$ are transformed into new functions. A general basis for the MI's involves a system of three coupled differential equations, which are equivalent to a single, third-order differential equation, whose solutions, as well known, are rather difficult to find. With the basis given in the previous section, the system comes out to be triangular with a 2×2 block. The coefficients of the associated homogeneous system read, in this basis:

$$f_{11}(x) = -\frac{2}{x}; \qquad\qquad f_{12}(x) = \frac{2}{x}; \qquad\qquad f_{13}(x) = 0; \tag{34}$$

$$f_{21}(x) = -\frac{1}{2x} + \frac{1}{3(x+1)} + \frac{1}{6(x-8)}; \quad f_{22}(x) = -\frac{1}{x+1} - \frac{1}{x-8}; \quad f_{23}(x) = 0; \tag{35}$$

$$f_{31}(x) = -\frac{1}{6x}; \qquad\qquad f_{32}(x) = \frac{1}{3}\left(1 + \frac{1}{x}\right); \qquad\qquad f_{33}(x) = -\frac{1}{x}; \tag{36}$$

while the known terms, related to the subtopologies, are:

$$N_1(x) \;=\; 0; \tag{37}$$

$$N_2(x) \;=\; \frac{1}{16x}\left[\frac{1}{2}H(0,0;x) - \frac{9}{4}H(0,-1;x) + \frac{1}{\sqrt{x(4-x)}}H(r,0;x) + \frac{\pi^2}{4}\right]$$

$$-\frac{1}{4(x+1)}\left[\frac{1}{9}H(0,0;x) - \frac{1}{2}H(0,-1;x) + \frac{5}{12\sqrt{x(4-x)}}H(r,0;x) + \frac{\pi^2}{18}\right]$$

$$-\frac{1}{8(x-8)}\left[\frac{1}{36}H(0,0;x) - \frac{1}{8}H(0,-1;x) - \frac{1}{3\sqrt{x(4-x)}}H(r,0;x) + \frac{\pi^2}{72}\right]; \tag{38}$$

$$N_3(x) \;=\; -\frac{1}{48x^2}\left[\frac{1}{2}H(-1,0,0;x) + H(r,r,0;x) + \frac{\pi^2}{4}H(-1;x)\right]. \tag{39}$$

The known terms are written in terms of generalized harmonic polylogarithms [10]. Since $f_{13}(x) = 0$ and $f_{23}(x) = 0$, F_3 decouples from the system of the first two MI's F_1 and F_2. It is therefore natural to compute F_1 and F_2 first. Once the latter are known, the third master integral F_3 can be computed by quadrature — i.e. by integration of the first-order differential equation (33) with $i = 3$.

7

5.1   Master Integrals Close to Zero External Momentum

Since $p_1^2 = p_2^2 = 0$,

$$q^2 = 2\,p_1 \cdot p_2 \tag{40}$$

and the limit

$$x \to 0 \tag{41}$$

is equivalent to the limit $p_1 \cdot p_2 \to 0$. However, limit (41) also implies the *stronger* limits $p_1^\mu \to 0$ and $p_2^\nu \to 0$, which reduce the vertex topology to a vacuum one. That can be seen for example by introducing Feynman parameters x_1, $x_2 \cdots x_5$ for the vertex amplitude and integrating over the loop momenta k_1^μ and k_2^ν. After that, the integrand can depend on relativistic invariants only, i.e. just on q^2. One then performs analytic continuation to Euclidean space, where the limit $q^2 \to 0$ implies $q^\mu \to 0$, i.e. $p_2^\mu = -p_1^\mu$. The diagram can then depend only on $p_1^2 = 0$ and therefore must be equal to that one computed for $p_1^\mu = 0$, i.e. to the corresponding vacuum amplitude. In particular, the MI's reduce to the following vacuum amplitudes:

$$F_1(x=0) = \int \frac{1}{(k_1^2 + m^2)\,(k_2^2 + m^2)\,k_1^2\,k_2^2\,[(k_1 - k_2)^2]^2}\,\mathcal{D}^4 k_1\,\mathcal{D}^4 k_2; \tag{42}$$

$$F_2(x=0) = \int \frac{1}{(k_1^2 + m^2)^2\,(k_2^2 + m^2)\,k_1^2\,k_2^2\,[(k_1 - k_2)^2]^2}\,\mathcal{D}^4 k_1\,\mathcal{D}^4 k_2; \tag{43}$$

$$F_3(x=0) = 0. \tag{44}$$

The above integrals can be computed exactly by using the identities

$$\frac{1}{k_i^2}\,\frac{1}{k_i^2 + m^2} = \frac{1}{m^2\,k_i^2} - \frac{1}{m^2\,(k_i^2 + m^2)}. \tag{45}$$

For our purposes, that is actually not necessary. By shrinking the massive lines in the infrared regions $k_1^2,\ k_2^2 \ll m^2$, we obtain:

$$F_1(x=0) \approx \frac{1}{m^4} \int^{\Lambda_{UV}} \frac{1}{k_1^2\,k_2^2\,[(k_1 - k_2)^2]^2}\,\mathcal{D}^4 k_1\,\mathcal{D}^4 k_2; \tag{46}$$

$$F_2(x=0) \approx \frac{1}{m^6} \int^{\Lambda_{UV}} \frac{1}{k_1^2\,k_2^2\,[(k_1 - k_2)^2]^2}\,\mathcal{D}^4 k_1\,\mathcal{D}^4 k_2, \tag{47}$$

where an ultraviolet cutoff $\Lambda_{UV} \approx m$ is assumed. The integrands above are invariant under the limit $\lambda \to 0$ in $D = 4$, where λ is introduced through the rescaling

$$k_1^\mu \to \lambda\,k_1^\mu, \tag{48}$$

$$k_2^\mu \to \lambda\,k_2^\mu. \tag{49}$$

All that implies an over-all logarithmic soft divergence[3]. There is also a logarithmic collinear singularity for $k_1^\mu \propto k_2^\mu$ related to the denominator $[(k_1 - k_2)^2]^2$. As a consequence[4]:

$$F_{1,2}(x) \sim \log^2 x, \qquad F_3(x) \sim 0 \qquad \text{for } x \to 0. \tag{50}$$

---

[3] There is not any soft sub-divergence, as can be seen by power-counting under the rescalings $k_1^\mu \to \lambda\,k_1^\mu$, $k_2^\mu \to k_2^\mu$ and $k_1^\mu \to k_1^\mu$, $k_2^\mu \to \lambda\,k_2^\mu$.

[4] If one sets $x = 0$ from the very beginning, $1/(D-4)$ poles of infrared nature do appear in F_1 and F_2.

That implies in particular that no pole terms $\approx 1/x$ can appear. As we are going to show explicitly, this qualitative information is sufficient for the solution of the system at small external momenta. Finally, let us note that we can set $m = 1$ from now on without loosing any information (m is an over-all scale).

6 Second-Order Differential Equation for $F_1(x)$

It is convenient to transform the system of differential equations for F_1 and F_2 into a single second-order equation for F_1 only. The procedure is standard: one takes an additional derivative with respect to x on both sides of eq. (33) with $i = 1$; dF_1/dx and dF_2/dx are replaced by the r.h.s. of eqs. (33) with $i = 1$ and $i = 2$ respectively; finally F_2 is replaced by its expression in terms of dF_1/dx and F_1, coming from the first of eqs. (33). We obtain:

$$\frac{d^2F_1}{dx^2}(x) + A(x)\frac{dF_1}{dx}(x) + B(x)F_1(x) + C(x) = 0, \tag{51}$$

where:

$$A(x) = \frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-8}; \tag{52}$$

$$B(x) = \frac{1}{x^2} + \frac{9}{8x} - \frac{4}{3(x+1)} + \frac{5}{24(x-8)}; \tag{53}$$

$$\begin{aligned}
C(x) = &-\left[\frac{1}{16x^2} - \frac{7}{128x} + \frac{1}{18(x+1)} - \frac{1}{1152(x-8)}\right]H(0,0;x) \\
&+\left[\frac{9}{32x^2} - \frac{63}{256x} + \frac{1}{4(x+1)} - \frac{1}{256(x-8)}\right]H(0,-1;x) \\
&-\left[\frac{1}{8x^2} - \frac{7}{32x} + \frac{5}{24(x+1)} + \frac{1}{96(x-8)}\right]\frac{1}{\sqrt{x(4-x)}}H(r,0;x) \\
&-\frac{\pi^2}{4}\left[\frac{1}{8x^2} - \frac{7}{64x} + \frac{1}{9(x+1)} - \frac{1}{576(x-8)}\right].
\end{aligned} \tag{54}$$

Let us make a few remarks:

1. the coefficient $A(x)$ of $F_1'(x)$ contains simple poles, while the coefficient $B(x)$ of $F_1(x)$ contains at most double poles. That is the necessary and sufficient condition on the differential equation to have regular singular points only [12];

2. the denominators entering the coefficients are:

$$\frac{1}{x}; \quad \frac{1}{x+1}; \quad \frac{1}{4-x}; \quad \frac{1}{x-8}. \tag{55}$$

The last denominator was not expected *a priori*, as it does not correspond to any threshold or pseudothreshold of the diagram. The denominator $1/(4-x)$ only appears in the known function $C(x)$ and it is related to the sub-topologies with the pseudothreshold in $s = -4m^2$;

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3. the known term $C(x)$ contains GHPL's (see appendix A) of weight at most two;

4. the GHPL $H(r, 0; x)$, containing a single square-root basic function, has a coefficient involving a square root $1/\sqrt{x(4-x)}$ to ensure reality of the solution across the pseudothreshold located at $x = 4$.

7 Power-Series Solution of the Second-Order Differential Equation for $F_1(x)$

The second-order differential equation (51) for F_1 is by far too complicated to be solved in a closed analytical form. We therefore look for solutions in the form of power series expansions centered around some points on the real axis. Let us recall that it is in any case necessary to consider the expansions around all the singular points, as the latter are by definition outside the convergence region of the ordinary expansions at regular points. Thanks to the differential equation satisfied by the considered functions, expansions around singular or regular points can be performed almost in the same way.

Since the coefficients A and B vanish for $x = 0, -1, 8$, we conclude that these points are singular points for the differential equation. Also $x = \infty$ is a singular point for the differential equation, as can be seen by changing variable to $y = 1/x$ and studying the limit $y \to 0$.

7.1 Expansion Around $x = 0$ — Small Momentum Expansion

In this section we consider the expansion of the master integral $F_1(x)$ around $x = 0$. Since the nearest singularity to the origin is located in $x = -1$, we expect a radius of convergence

$$R_0 = 1, \tag{56}$$

i.e. convergence in the complex x-plane for

$$|x| < 1. \tag{57}$$

For real x, that means:

$$-1 < x < 1. \tag{58}$$

Because of linearity, the general solution of the complete inhomogeneous equation can be written as the superposition of the *general* solution of the associated *homogeneous* equation — i.e. eq. (51) with $C(x) = 0$,

$$\frac{d^2 F_1^{(0)}}{dx^2}(x) + A(x)\frac{dF_1^{(0)}}{dx}(x) + B(x)F_1^{(0)}(x) = 0, \tag{59}$$

plus a *particular* solution of the *complete* equation:

$$F_1(x) = F_1^{(0)}(x) + \bar{F}_1(x). \tag{60}$$

We will use systematically eq. (60) in the following expansions.

7.1.1 Homogeneous Equation

Let us first consider the associated homogeneous equation. Since zero is a "singular regular point", we can look for solutions having the form of a singular function in $x = 0$, $S(x)$, multiplied by a regular (i.e. convergent) power-series[5] :

$$F_1^{(0)}(x) = S(x) \sum_{n=0}^{\infty} A_n x^n, \tag{61}$$

where A_n are numerical coefficients determined from the differential equation itself and from some initial or boundary conditions. The function $S(x)$, giving the leading singularity for $x \to 0$, is assumed to be of power-like form

$$S(x) = x^{\alpha}, \tag{62}$$

and solves the limit of the homogeneous equation for $x \to 0$:

$$S''(x) + \frac{3}{x} S'(x) + \frac{1}{x^2} S(x) = 0. \tag{63}$$

By inserting a solution of the form (62) we obtain the indicial equation $(\alpha + 1)^2 = 0$ with a double zero in $\alpha = -1$, implying two independent pre-factors of the form:

$$S(x) = \frac{1}{x}, \quad \frac{\log x}{x}. \tag{64}$$

The most general solution of the homogeneous differential equation is therefore of the form:

$$F_1^{(0)}(x) = \sum_{n=-1}^{\infty} a_n x^n + \log x \sum_{n=-1}^{\infty} b_n x^n, \tag{65}$$

where we have absorbed a $1/x$ factor inside the series by defining:

$$a_n \equiv A_{n+1}. \tag{66}$$

By expanding the differential equation (59) around $x = 0$ and substituting the series representation in eq. (65), we can obtain recursively all the desired coefficients. The first few are:

$$a_0 = -\frac{1}{4}a_{-1} - \frac{3}{8}b_{-1}; \qquad\qquad b_0 = -\frac{1}{4}b_{-1}; \tag{67}$$

$$a_1 = \frac{5}{32}a_{-1} + \frac{33}{128}b_{-1}; \qquad\qquad b_1 = \frac{5}{32}b_{-1}; \tag{68}$$

$$a_2 = -\frac{7}{64}a_{-1} - \frac{25}{128}b_{-1}; \qquad\qquad b_2 = -\frac{7}{64}b_{-1}; \tag{69}$$

$$a_3 = \frac{173}{2048}a_{-1} + \frac{2561}{16384}b_{-1}; \qquad\qquad b_3 = \frac{173}{2048}b_{-1}. \tag{70}$$

Let us make a few remarks:

---

[5] This case is to be contrasted to that of an "irregular singular point", in which the singularity of the differential equation is so strong that no factorization of the singularity of the form (61) is possible. In the latter case, one typically obtains series with an infinite number of negative powers or asymptotic (divergent) expansions.

- setting to zero for example a_{-1} and b_{-1}, all the higher-order coefficient vanish and we obtain the trivial solution: that is the homogeneity property;

- there is a triangular structure: all the coefficients b_i are proportional to the lowest-order one b_{-1}, implying that by setting $b_{-1} = 0$ we obtain a solution without the series with the logarithmic prefactor and with the simple pole in $x = 0$ only. The coefficients a_i depend instead on both a_{-1} and b_{-1}, implying that by setting $a_{-1} = 0$ one obtain a solution containing both serieses;

- A first independent solution can be obtained by taking for example $a_{-1} = 1$ and $b_{-1} = 0$, resulting in a function without logarithmic terms. A second independent solution can be obtained by taking $a_{-1} = 0$ and $b_{-1} = 1$, resulting in a function having both the power and the logarithmic terms. With a pictorial language, we may say that the logarithmic series "feeds" the standard one, while the vice-versa is not true.

- The singularity of the differential equation at $x = 0$ produces a logarithmic branch point in the solution at $x = 0$ whenever $b_{-1} \neq 0$.

7.1.2 Complete Equation

Let us now consider the complete equation (51). We have to expand around $x = 0$ also the inhomogeneous term $C(x)$, which is known (see appendix B):

$$C(x) = \frac{1}{x^2} \left[\sum_{n=0}^{\infty} k_n x^n + \log x \sum_{n=0}^{\infty} q_n x^n + \log^2 x \sum_{n=0}^{\infty} r_n x^n \right]. \tag{71}$$

The known term has a double pole in $x = 0$, multiplied also by a single or a double logarithm of x. The expected radius of convergence of the multiplying series is one:

$$R = 1. \tag{72}$$

The explicit expressions of the lowest-order coefficients read:

$$k_0 = \frac{1}{8} - \frac{\pi^2}{32} \,; \qquad\qquad q_0 = -\frac{1}{16} \,; \qquad r_0 = -\frac{1}{32} \,; \tag{73}$$

$$k_1 = \frac{23}{288} + \frac{7}{256}\pi^2 \,; \qquad\qquad q_1 = \frac{19}{192} \,; \qquad r_1 = \frac{7}{256} \,; \tag{74}$$

$$k_2 = -\frac{3931}{28800} - \frac{57}{2048}\pi^2 \,; \qquad q_2 = -\frac{671}{7680} \,; \qquad r_2 = -\frac{57}{2048} \,; \tag{75}$$

$$k_3 = \frac{1789247}{11289600} + \frac{455}{16384}\pi^2 \,; \qquad q_3 = \frac{38791}{430080} \,; \qquad r_3 = \frac{455}{16384} \,. \tag{76}$$

Let us look for a particular solution of (51) of the form:

$$\bar{F}_1(x) = \sum_{n=-1}^{\infty} p_n x^n + \log x \sum_{n=-1}^{\infty} q_n x^n + \log^2 x \sum_{n=0}^{\infty} c_n x^n \,. \tag{77}$$

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By inserting the above form of the solution into the differential equation expanded around $x = 0$ with the known term $C(x)$ given by the series expansion in eq. (71), we obtain for the coefficients:

$$p_0 = -\frac{1}{4}p_{-1} - \frac{3}{8}q_{-1} + \frac{\pi^2}{32} - \frac{1}{16} ; \tag{78}$$

$$q_0 = -\frac{1}{4}q_{-1} - \frac{1}{16} ; \tag{79}$$

$$c_0 = \frac{1}{32} ; \tag{80}$$

$$p_1 = \frac{5}{32}p_{-1} + \frac{33}{128}q_{-1} - \frac{\pi^2}{64} + \frac{5}{576} ; \tag{81}$$

$$q_1 = \frac{5}{32}q_{-1} + \frac{1}{96} ; \tag{82}$$

$$c_1 = -\frac{1}{64} ; \tag{83}$$

$$p_2 = -\frac{7}{64}p_{-1} - \frac{25}{128}q_{-1} + \frac{13}{1152}\pi^2 - \frac{1093}{518400} ; \tag{84}$$

$$q_2 = -\frac{7}{64}q_{-1} - \frac{121}{17280} ; \tag{85}$$

$$c_2 = \frac{13}{1152} ; \tag{86}$$

$$p_3 = \frac{173}{2048}p_{-1} + \frac{2561}{16384}q_{-1} - \frac{5\pi^2}{576} + \frac{649}{635040} ; \tag{87}$$

$$q_3 = \frac{173}{2048}q_{-1} + \frac{95}{24192} ; \tag{88}$$

$$c_3 = -\frac{5}{576} ; \tag{89}$$

$$p_4 = -\frac{563}{8192}p_{-1} - \frac{42631}{327680}q_{-1} + \frac{407}{57600}\pi^2 - \frac{3217}{6720000} ; \tag{90}$$

$$q_4 = -\frac{563}{8192}q_{-1} - \frac{1141}{432000} ; \tag{91}$$

$$c_4 = \frac{407}{57600} . \tag{92}$$

The coefficients c_i of the double-logarithmic terms are completely determined, while the remaining ones p_i and q_i are fixed once two arbitrary coefficients, such as for example p_{-1} and q_{-1}, have been fixed. That is exactly the same arbitrariness that we have already found for the associated homogeneous equation. Since the general solution of the latter has already been found, we need only to find a particular, i.e. a single solution of the complete equation. Let us choose for example:

$$p_{-1} = 0 \quad \text{and} \quad q_{-1} = 0 . \tag{93}$$

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With the following *arbitrary* choice, the particular solution does not contain any pole term and its numerical coefficients are completely fixed:

$$p_0 = -\frac{1}{16} + \frac{\pi^2}{32}; \qquad\qquad q_0 = -\frac{1}{16}; \qquad\qquad c_0 = \frac{1}{32}; \qquad (94)$$

$$p_1 = \frac{5}{576} - \frac{\pi^2}{64}, \qquad\qquad q_1 = \frac{1}{96}; \qquad\qquad c_1 = -\frac{1}{64}, \qquad (95)$$

$$p_2 = -\frac{1093}{518400} + \frac{13\pi^2}{1152}; \qquad q_2 = -\frac{121}{17280}, \qquad c_2 = \frac{13}{1152}; \qquad (96)$$

$$p_3 = \frac{649}{635040} - \frac{5\pi^2}{576}, \qquad q_3 = \frac{95}{24192}; \qquad c_3 = -\frac{5}{576}, \qquad (97)$$

$$p_4 = -\frac{3217}{6720000} + \frac{407\pi^2}{57600}; \qquad q_4 = -\frac{1141}{432000}, \qquad c_4 = \frac{407}{57600}. \qquad (98)$$

The general solution of the complete equation is therfore:

$$F_1(x) = \sum_{n=-1}^{\infty} a_n x^n + \log x \sum_{n=-1}^{\infty} b_n x^n + \sum_{n=0}^{\infty} p_n x^n + \log x \sum_{n=0}^{\infty} q_n x^n + \log^2 x \sum_{n=0}^{\infty} c_n x^n. \quad (99)$$

To uniquely determine the solution, one has to impose some boundary or initial conditions. As already discussed in the previous section, $F_1(x)$ can have at most a logarithmic singularity for $x \to 0$, implying that the coefficients of the power singularities must vanish:

$$a_{-1} = 0, \quad b_{-1} = 0; \qquad (100)$$

and the solution of the homogeneous equation to be selected, reduces to the trivial one. The particular solution $\bar{F}_1$ of the complete equation that we have chosen is therefore the expansion of the master integral F_1 as defined by eq. (28):

$$F_1(x) = \sum_{n=0}^{\infty} p_n x^n + \log x \sum_{n=0}^{\infty} q_n x^n + \log^2 x \sum_{n=0}^{\infty} c_n x^n, \qquad (101)$$

where the coefficients are given in eqs. (94-98)

Let us stress that we have been able to obtain the complete analytical expression of the coefficients of this small-momentum expansion because of the knowledge of the MI at small momentum transferred. As we are going to show in the next sections, the absence of a similar knowledge in other expansion points will limit us to a numerical estimate of the corresponding coefficients.

As expected from the threshold structure, the solution given in eq. (101) is real for $x > 0$ (space-like region) and has a non-vanishing imaginary part for $x < 0$ (time-like region). The latter can be determined by simply giving the prescription for the $\log x$ factor. Since $s \to s + i\epsilon$ with $\epsilon = +0$, eq. (31) gives for $x < 0$:

$$x \to -|x| - i\epsilon. \qquad (102)$$

Consequently, we have:

$$\begin{aligned} \log x &\to \log|x| - i\pi; \\ \log^2 x &\to \log^2|x| - \pi^2 - 2i\pi \log|x|. \end{aligned} \qquad (103)$$

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Therefore $F_1(x)$ becomes complex for $x < 0$ with:

$$\mathrm{Re}\, F_1(x) \;=\; \sum_{n=0}^{\infty} p_n x^n + \log|x| \sum_{n=0}^{\infty} q_n x^n + \left(\log^2|x| - \pi^2\right) \sum_{n=0}^{\infty} c_n x^n \,; \tag{104}$$

$$\mathrm{Im}\, F_1(x) \;=\; -\pi \left(\sum_{n=0}^{\infty} q_n\, x^n + 2\log|x| \sum_{n=0}^{\infty} c_n\, x^n \right). \tag{105}$$

In the following sections, we will "move" on the real axis constructing the solution from $x = 0$ to $x = \infty$; then with an analytic continuation we will switch to $x = -\infty$ and finally move back to $x = 0$ through negative values of x.

7.1.3 Improved Expansion — Bernoulli Variable

A series for F_1 with better convergence properties than the previous one in x can be constructed by expanding in the Bernoulli variable [20]

$$t \;\equiv\; \log\left(8\,\frac{1+x}{8-x} \right) = \frac{9}{8}x - \frac{63}{128}x^2 + \frac{171}{512}x^3 - \frac{4095}{16384}x^4 + \frac{32769}{163840}x^5 + \mathcal{O}\left(x^6\right) \tag{106}$$

$$= \; 1.125x - 0.492187x^2 + 0.333984x^3 - 0.249938x4 + 0.200006x^5 + \mathcal{O}\left(x^6\right).$$

Eq. (106) realizes a one-to-one map between $x \in (-1, 8)$ and $t \in (-\infty, \infty)$. Furthermore,

$$t = 0 \;\iff\; x = 0, \tag{107}$$

while

$$\begin{aligned} t &\to -\infty &&\text{for} && x \to -1^+ \\ t &\to +\infty &&\text{for} && x \to +8^-. \end{aligned} \tag{108}$$

t diverges logarithmically when x approaches the singularities of the differential equation closest to the origin. Since this variable "follows" the singularities of the differential equation, we expect a faster convergence with the order of truncation in t rather than in x. The inverse of eq. (106) reads:

$$x = \frac{8(e^t - 1)}{e^t + 8} = \sum_{n=1}^{\infty} c_n\, t^n. \tag{109}$$

The first few terms are:

$$x = \frac{8}{9}t + \frac{28}{81}t^2 + \frac{44}{729}t^3 - \frac{35}{6561}t^4 - \frac{1733}{295245}t^5 + O\left(t^6\right) \tag{110}$$

$$= \; 0.888888t + 0.345679\, t^2 + 0.0603566\, t^3 - 0.00533455\, t^4 - 0.00586970\, t^5 + O\left(t^6\right).$$

The radius of convergence of the above series is determined by looking at the singularities of $x = x(t)$ in the complex t-plane, located in

$$t_k = 3\log 2 + i(2k+1)\pi, \tag{111}$$

where k is an integer. The closest singularities to the origin, for $k = 0, -1$, give

$$r_0 = \sqrt{\pi^2 + 9 \log^2 2} \approx 3.76745. \tag{112}$$

We now substitute the r.h.s. of eq. (109) in the series expansion for $F_1(x)$ obtained previously and finally expand in powers of t, to have[6]:

$$\tilde{F}_1(t) \equiv F_1(x(t)) = \sum_{n=0}^{\infty} \alpha_n t^n + \log x \sum_{n=0}^{\infty} \beta_n t^n + \log^2 x \sum_{n=0}^{\infty} \gamma_n t^n, \tag{113}$$

where the coefficients α_n, β_n and γ_n are determined from a_n, b_n and c_n respectively. The first few orders read:

$$\alpha_0 = -\frac{1}{16} + \frac{\pi^2}{32}; \qquad \beta_0 = -\frac{1}{16}; \qquad \gamma_0 = \frac{1}{32}, \tag{114}$$

$$\alpha_1 = \frac{5}{648} - \frac{\pi^2}{72}; \qquad \beta_1 = \frac{1}{108}, \qquad \gamma_1 = -\frac{1}{72}; \tag{115}$$

$$\alpha_2 = \frac{3503}{2624400} + \frac{41}{11664}\pi^2, \qquad \beta_2 = -\frac{169}{87480}; \qquad \gamma_2 = \frac{41}{11664}. \tag{116}$$

We have computed $\tilde{F}_1(t)$ as a function of t (eq. (113)) by substituting the series (109) into the series (101). The problem now is that of computing the radius of convergence ρ_0 of the series expansion for $\tilde{F}_1(t)$. The "internal" series for $x = x(t)$ converges inside the circle in the t-plane of radius r_0 given in eq. (112); the "external" series for $F(x)$ converges in the unitary circle $|x| < 1$ in the x-plane (eq. (57)). As well known, power series always converge inside circles. The point is that a circle in the t-plane in general is *not* mapped by the function $x = x(t)$ into a circle in the x-plane. Our problem has the following geometrical formulation: one has to find the largest circle in the t-plane satisfying $|t| < r_0$, which is mapped inside the circle of unitary radius in x-plane by the transformation $x = x(t)$ [21]. On the unitary circle $x = \exp i\varphi$ and one has to look for a minimum over φ of

$$|t| = \left| \log \left(8 \frac{1 + \exp i\varphi}{8 - \exp i\varphi} \right) \right| \geq \log \frac{16}{7} < r_0, \tag{117}$$

where the minimum is obtained for $\varphi = 0$. We then find[7]:

$$\rho_0 = \log \frac{16}{7} \approx 0.826679. \tag{120}$$

---

[6] Let us note that we did not express the logarithmic pre-factors $\log x$ and $\log^2 x$ as series in t, because there was no practical advantage for doing that.

[7] We have shown that the convergence radius of the t-series of F_1 is *not smaller* than ρ_0, but it can actually be larger, depending on possible elimination of singularities. That can be illustrated with the following (rather trivial) example. Let us consider the expansion around $x = 0$ of a differential equation having a solution of the form

$$\phi(x) = \log \frac{8(1 + x)}{8 - x}. \tag{118}$$

If we go to the Bernoulli variable t defined in eq. (106), we obtain

$$\tilde{\phi}(t) = \phi(x(t)) = t, \tag{119}$$

which can be analytically continued to all the t-plane, implying an infinite radius of convergence.

In general, for a series expansion centered in x_0 with nearest singularities a and b satisfying

$$a \;<\; x_0 \;<\; b, \tag{121}$$

the Bernoulli variable t is defined as:

$$t \;=\; \log\left(\frac{b - x_0}{x_0 - a}\frac{x - a}{b - x}\right). \tag{122}$$

Note that

$$t \;=\; 0 \quad\Longleftrightarrow\quad x \;=\; x_0, \tag{123}$$

while

$$\begin{aligned} t &\;\to\; -\infty \quad \text{for} \quad x \to a^+ \,; \\ t &\;\to\; +\infty \quad \text{for} \quad x \to b^-. \end{aligned} \tag{124}$$

The following particular case are relevant in the following:

- $x_0 \to \infty$:

$$t \;=\; \log\left(\frac{x - a}{x - b}\right) \,; \tag{125}$$

- $b \to \infty$:

$$t \;=\; -\log\left(\frac{x - a}{x_0 - a}\right) \,; \tag{126}$$

- $a \to -\infty$:

$$t \;=\; -\log\left(\frac{x - b}{x_0 - b}\right) \,. \tag{127}$$

7.2 Expansion Around $x = 8$

In this section we consider the expansion of $F_1(x)$ around the (space-like) point $x = 8$, which is a regular singular point of the differential equation not corresponding to any threshold or pseudothreshold of the diagram. The singular point of $F_1(x)$ closest to $x = 8$ is located in $x = 0$. We then expect the expansion around $x = 8$ to have a radius of convergence

$$R_8 \;=\; 8, \tag{128}$$

so that the series converges in the complex x-plane for

$$|x - 8| \;<\; 8. \tag{129}$$

For real x, that means:

$$0 \;<\; x \;<\; 16. \tag{130}$$

7.2.1 Homogeneous Equation

By solving the indicial equation as in the previous section, we obtain a double zero in zero, so that the homogeneous equation has a solution of the form:

$$F_1^{(0)}(x) = \sum_{n=0}^{\infty} a_n(x-8)^n + \log(x-8)\sum_{n=0}^{\infty} b_n(x-8)^n. \tag{131}$$

The coefficients are, of course, different from those ones of the previous section — we use the same notation to avoid introducing too many symbols. The first few coefficients read:

$$a_1 = -\frac{5}{24}\,a_0 - \frac{5}{72}\,b_0\,; \qquad\qquad b_1 = -\frac{5}{24}\,b_0\,; \tag{132}$$

$$a_2 = \frac{59}{1728}\,a_0 + \frac{187}{10368}\,b_0\,; \qquad\qquad b_2 = \frac{59}{1728}\,b_0\,; \tag{133}$$

$$a_3 = -\frac{635}{124416}\,a_0 - \frac{7561}{2239488}\,b_0\,; \qquad\qquad b_3 = -\frac{635}{124416}b_0\,; \tag{134}$$

$$a_4 = \frac{2171}{2985984}\,a_0 + \frac{59447}{107495424}\,b_0\,; \qquad\qquad b_4 = \frac{2171}{2985984}\,b_0\,. \tag{135}$$

We have again a triangular structure of the coefficients, as in the previous case.

7.2.2 Complete Equation

The expansion of the known term around $x = 8$ is of the following form:

$$C(x) = \sum_{n=-1}^{\infty} q_n(x-8)^n\,, \tag{136}$$

where the first three coefficients q_n are:

$$q_{-1} = \frac{1}{2304}\pi^2 + \frac{1}{768}M_0\sqrt{2} - \frac{1}{256}M_1 + \frac{1}{192}M_2\sqrt{2}\log 2 + \frac{1}{256}\log^2 2\,; \tag{137}$$

$$\begin{aligned} q_0 = {} & -\frac{13}{82944}\pi^2 - \frac{29}{55296}M_0\sqrt{2} + \frac{13}{9216}M_1 - \frac{29}{13824}M_2\sqrt{2}\log 2 + \frac{1}{768}\log 2 \\ & -\frac{13}{9216}\log^2 2 - \frac{1}{1024}\log 3\,; \end{aligned} \tag{138}$$

$$\begin{aligned} q_1 = {} & \frac{451}{11943936}\pi^2 + \frac{4637}{31850496}M_0\sqrt{2} - \frac{451}{1327104}M_1 + \frac{4637}{7962624}M_2\sqrt{2}\log 2 \\ & -\frac{551}{884736}\log 2 + \frac{451}{1327104}\log^2 2 + \frac{61}{147456}\log 3\,. \end{aligned} \tag{139}$$

We have defined the following transcendental constants:

$$
\begin{aligned}
M_0 \;&\equiv\; \int_0^1 \frac{\log(1+y)}{\sqrt{y(1+y)}}\,dy \\
&=\; \frac{\pi^2}{6} + 4\log 2\log(\sqrt{2}-1) + 2\log^2(\sqrt{2}-1) + 4\mathrm{Li}_2\!\left[i(\sqrt{2}-1)\right] + 4\mathrm{Li}_2\!\left[-i(\sqrt{2}-1)\right] \\
&\approx\; 0.425435\,, \tag{140}
\end{aligned}
$$

$$
M_1 \;=\; \frac{\pi^2}{6} + \frac{9}{2}\log^2 2 + \mathrm{Li}_2\!\left(-\frac{1}{8}\right) \approx 3.68568\,; \tag{141}
$$

$$
M_2 \;=\; \log(1+\sqrt{2}) \approx 0.881374\,. \tag{142}
$$

The general solution of the inhomogeneous equation reads:

$$
\bar{F}_1(x) \;=\; \sum_{n=0}^{\infty} r_n (x-8)^n + \log(x-8)\sum_{n=0}^{\infty} p_n (x-8)^n\,, \tag{143}
$$

with the first three terms given by:

$$
\begin{aligned}
r_1 \;=\;& -\frac{1}{2304}\pi^2 - \frac{1}{768}M_0\sqrt{2} + \frac{1}{256}M_1 - \frac{1}{192}M_2\sqrt{2}\log 2 - \frac{1}{256}\log^2 2 \\
& -\frac{5}{24}r_0 - \frac{5}{72}p_0\,; \tag{144}
\end{aligned}
$$

$$
p_1 \;=\; -\frac{5}{24}p_0\,; \tag{145}
$$

$$
\begin{aligned}
r_2 \;=\;& \frac{19}{165888}\pi^2 + \frac{79}{221184}M_0\sqrt{2} - \frac{19}{18432}M_1 + \frac{79}{55296}M_2\sqrt{2}\log 2 - \frac{1}{3072}\log 2 \\
& +\frac{19}{18432}\log^2 2 + \frac{1}{4096}\log 3 + \frac{59}{1728}r_0 + \frac{187}{10368}p_0\,; \tag{146}
\end{aligned}
$$

$$
p_2 \;=\; \frac{59}{1728}p_0\,. \tag{147}
$$

A particular solution can be obtained by setting $p_0 = 0$, which makes all the coefficients p_n vanishing: that way the series multiplied by the logarithm disappears from $\bar{F}_1$. One can also set $r_0 = 0$. The first three coefficients are then given by:

$$
r_0 \;=\; 0\,; \tag{148}
$$

$$
p_0 \;=\; 0\,; \tag{149}
$$

$$
r_1 \;=\; -\frac{1}{2304}\pi^2 - \frac{1}{768}M_0\sqrt{2} + \frac{1}{256}M_1 - \frac{1}{192}M_2\sqrt{2}\log 2 - \frac{1}{256}\log^2 2\,; \tag{150}
$$

$$
p_1 \;=\; 0\,; \tag{151}
$$

$$
\begin{aligned}
r_2 \;=\;& \frac{19}{165888}\pi^2 + \frac{79}{221184}M_0\sqrt{2} - \frac{19}{18432}M_1 + \frac{79}{55296}M_2\sqrt{2}\log 2 - \frac{1}{3072}\log 2 \\
& +\frac{19}{18432}\log^2 2 + \frac{1}{4096}\log 3\,; \tag{152}
\end{aligned}
$$

$$
p_2 \;=\; 0\,. \tag{153}
$$

The general solution of the differential equation is given by

$$F_1(x) = F_1^{(0)}(x) + \bar{F}_1(x), \tag{154}$$

and depends on the arbitrary constants a_0 and b_0 entering $F_1^{(0)}(x)$. Since we know from general arguments that $x = 8$ is a regular point for the solution, we can impose the logarithmic series be absent by requiring

$$b_0 = 0. \tag{155}$$

This condition gives:

$$F_1(x) = \sum_{n=0}^{\infty} s_n (x-8)^n, \tag{156}$$

where:

$$s_0 = a_0; \tag{157}$$

$$s_1 = -\frac{5}{24} a_0 + r_1; \tag{158}$$

$$s_2 = \frac{59}{1728} a_0 + r_2, \tag{159}$$

etc.. Then, the expansion of $F_1(x)$ around $x = 8$ given in eq. (156) does not determine $F_1(x)$ uniquely, because it still contains the free parameter a_0. By using the matching procedure described in sect. 7.2.4, we obtain the following numerical estimate for this coefficient:

$$a_0 = 0.0321062814000779405116. \tag{160}$$

7.2.3 Improved Expansion

In order to improve the convergence of the series so far obtained, we move from the series in x to the one in the Bernoulli variable

$$t \equiv \log \frac{x}{8}, \tag{161}$$

with the inverse:

$$x = 8e^t = 8 + 8t + 4t^2 + \frac{4}{3}t^3 + \frac{1}{3}t^4 + \frac{1}{15}t^5 + \frac{1}{90}t^6 + \mathcal{O}(t^7). \tag{162}$$

Unlike the previous case ($x = 0$), the above series has an infinite radius of convergence:

$$r_8 = \infty. \tag{163}$$

Substituting the r.h.s. of the above equation in the series expansion for $F_1(x)$ obtained previously and, finally, expanding in powers of t, we have:

$$\tilde{F}_1(t) \equiv F_1(x(t)) = \sum_{n=0}^{\infty} \alpha_n t^n, \tag{164}$$

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where the first three coefficients α_n are:

$$\alpha_1 = 8\,s_0 \, ; \tag{165}$$

$$\alpha_2 = \frac{73}{54}a_0 + \frac{29}{5184}\pi^2 + \frac{61}{3456}M_0\sqrt{2} - \frac{29}{576}M_1 + \frac{61}{864}M_2\sqrt{2}\log 2 - \frac{1}{48}\log 2$$
$$+ \frac{29}{576}\log^2 2 + \frac{1}{64}\log 3 \, ; \tag{166}$$

$$\alpha_3 = -\frac{343}{486}a_0 - \frac{203}{46656}\pi^2 - \frac{929}{62208}M_0\sqrt{2} + \frac{203}{5184}M_1 - \frac{929}{15552}M_2\sqrt{2}\log 2$$
$$+ \frac{7}{192}\log 2 - \frac{203}{5184}\log^2 2 - \frac{7}{288}\log 3 \, . \tag{167}$$

A computation of the convergence radius ρ_8 of the series in t analogous to the one of the previous section gives:

$$\rho_8 = \log 2 \approx 0.693147. \tag{168}$$

7.2.4 The Matching Condition

a_0 can be computed by imposing that the series in eq. (101), which is completely determined, and that one in eq. (156) assume the *same* value in a given point in the intersection of the respective domains of convergence. One has to take a point lying in the interval

$$-1 < x < 1, \tag{169}$$

where the series of eq. (101) converges, as well as in the interval

$$0 < x < 16, \tag{170}$$

where the series in eq. (156) converges. One has therefore to choose a point in the interval

$$0 < x < 1, \tag{171}$$

such as for example $x = 1/2$. If we deal with infinite series, this procedure exactly determines the coefficient a_0. As we have shown above, however, we can only determine an arbitrary, but finite, number of coefficients of both serieses and the matching has to be made in an approximate numerical way by using truncated series. The number of terms of the two serieses that must be computed depend on the required precision on a_0. Our goal is to give $F_1(x)$ with a relative precision of better then 10^{-15} (double precision) using a relatively small number of terms in the serieses (around 30). The problem is that any point in the interval (171) is close to the boundary of the convergence domain for the series centered around $x = 8$, where convergence is slow, implying that many terms are needed for high accuracy. In other words, a direct matching between the series in eq. (101) and that one in eq. (156) is not the good algorithm. As we have shown before, a first improvement is obtained by re-writing the series expansions in terms of the relevant Bernoulli variables. But it is convenient to use additional series expansions centered around auxiliary regular points in the range $0 < x < 8$, such as for example $x = 3$. This topic will be discussed in more detail in sect. 7.5.

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7.3 Expansion Around $x = \infty$ — Large Momentum Expansion

In this section we consider the expansion of $F_1(x)$ around $x = \infty$, which generates a large-momentum expansion. Since the closest singularity to $x = \infty$ is at $x = 8$, we expect the expansion around infinity to be convergent outside the circle of radius 8, i.e. for

$$|x| > 8. \tag{172}$$

The expansion at infinity is studied systematically by changing variable to

$$y \equiv \frac{1}{x} \tag{173}$$

and considering the limit

$$y \to 0. \tag{174}$$

7.3.1 Homogeneous Equation

Let us write the solution as usual as:

$$F_1^{(0)}(y) = S(y) \sum_{n=0}^{\infty} A_n y^n. \tag{175}$$

The pre-factor is assumed to have power-like form,

$$S(y) = y^{\beta}, \tag{176}$$

and to be a solution of the homogeneous equation in the limit (174):

$$S''(y) - \frac{3}{y} S'(y) + \frac{4}{y^2} S(y) = 0. \tag{177}$$

The indicial equation is $(\beta - 2)^2 = 0$, with a double zero in $\beta = 2$ and therefore the solution is, in the original x variable, of the form:

$$F_1^{(0)}(x) = \sum_{n=2}^{\infty} \frac{a_n}{x^n} + \log x \sum_{n=2}^{\infty} \frac{b_n}{x^n}. \tag{178}$$

Because of the presence of the logarithmic term in front of the second series, the infinity is a singular point — more exactly, a branch point of infinite order. Furthermore, since positive powers of x do not appear, $x = \infty$ is a regular point for the serieses above. Let us write a bunch of coefficients in terms of the lowest-order ones a_2 and b_2:

$$a_3 = 2\,a_2 - 3\,b_2\,; \qquad\qquad\qquad b_3 = 2\,b_2\,; \tag{179}$$

$$a_4 = 10\,a_2 - \frac{33}{2}\,b_2\,; \qquad\qquad\qquad b_4 = 10\,b_2\,; \tag{180}$$

$$a_5 = 56\,a_2 - 100\,b_2\,; \qquad\qquad\qquad b_5 = 56\,b_2\,; \tag{181}$$

$$a_6 = 346\,a_2 - \frac{2561}{4}\,b_2\,; \qquad\qquad\qquad b_6 = 346\,b_2\,; \tag{182}$$

$$a_7 = 2252\,a_2 - \frac{42631}{10}\,b_2\,; \qquad\qquad\qquad b_7 = 2252\,b_2\,. \tag{183}$$

Let us notice the power-like growth of the coefficients, which should scale asymptotically as $\approx 8^n$ (b_7/b_6 is already ≈ 6.5).

7.3.2 Complete Equation

The non-trivial part of the expansion of the known term $C(x)$ around $x = \infty$ is related to the expansion of the GHPL's around this point, which is discussed in appendix B. The expansion is of the form:

$$C(x) = \sum_{n=4}^{\infty} \frac{k_n}{x^n} + \log x \sum_{n=4}^{\infty} \frac{l_n}{x^n} + \log^2 x \sum_{n=4}^{\infty} \frac{m_n}{x^n} \qquad (|x| > 8), \qquad (184)$$

where the lowest-order coefficients read:

$$k_4 = \frac{\pi^2}{48}; \qquad\qquad l_4 = 0; \qquad\qquad m_4 = -\frac{7}{16}; \qquad (185)$$

$$k_5 = \frac{1}{2} + \frac{13\pi^2}{48}; \qquad l_5 = -\frac{7}{4}; \qquad m_5 = -\frac{43}{16}; \qquad (186)$$

$$k_6 = \frac{1}{8} + \frac{125\pi^2}{48}; \qquad l_6 = -\frac{131}{8}; \qquad m_6 = -\frac{331}{16}; \qquad (187)$$

$$k_7 = -\frac{40}{9} + \frac{357\pi^2}{16}; \qquad l_7 = -\frac{3437}{24}; \qquad m_7 = -\frac{2569}{16}; \qquad (188)$$

$$k_8 = -\frac{18323}{288} + \frac{8827\pi^2}{48}; \qquad l_8 = -\frac{56953}{48}; \qquad m_8 = -\frac{20301}{16}. \qquad (189)$$

Since the differential equation involves a second derivative and the known term has double-logarithmic terms, the solution must contain up to four powers of the logarithm:

$$\bar{F}_1(x) = \sum_{n=2}^{\infty} \frac{p_n}{x^n} + \log x \sum_{n=2}^{\infty} \frac{q_n}{x^n} + \log^2 x \sum_{n=2}^{\infty} \frac{r_n}{x^n} + \log^3 x \sum_{n=2}^{\infty} \frac{s_n}{x^n} + \log^4 x \sum_{n=2}^{\infty} \frac{t_n}{x^n}. \qquad (190)$$

Substituting the above form for the solution into the equation, we obtain the following relations among the coefficients:

$$r_2 = -\frac{\pi^2}{96}; \qquad (191)$$

$$s_2 = 0; \qquad (192)$$

$$t_2 = \frac{7}{192}; \qquad (193)$$

$$p_3 = 2\,p_2 - 3\,q_2 - \frac{5}{48}\pi^2 + \frac{27}{8}; \qquad (194)$$

$$q_3 = 2\,q_2 + \frac{\pi^2}{16} + \frac{9}{8}; \qquad (195)$$

$$r_3 = -\frac{13}{16} - \frac{\pi^2}{48}; \qquad (196)$$

$$s_3 = -\frac{7}{16}; \qquad (197)$$

$$t_3 = \frac{7}{96}. \qquad (198)$$

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The values of p_2 and q_2 are left undetermined by the differential equation and a particular solution can be found by imposing for instance

$$p_2 = 0; \qquad q_2 = 0, \tag{199}$$

which we assume from now on. The general solution is given by:

$$
\begin{aligned}
F_1(x) &= F_1^{(0)}(x) + \bar{F}_1(x) \\
&= \sum_{n=2}^{\infty} \frac{\tilde{p}_n}{x^n} + \log x \sum_{n=2}^{\infty} \frac{\tilde{q}_n}{x^n} + \log^2 x \sum_{n=2}^{\infty} \frac{r_n}{x^n} + \log^3 x \sum_{n=2}^{\infty} \frac{s_n}{x^n} + \log^4 x \sum_{n=2}^{\infty} \frac{t_n}{x^n},
\end{aligned}
\tag{200}
$$

where we have defined:

$$\tilde{p}_n \equiv p_n + a_n; \qquad \tilde{q}_n \equiv q_n + b_n. \tag{201}$$

Let us note that the coefficients of the double, triple and fourth logarithm are uniquely determined because the associated homogeneous equation has no terms of this form: we therefore obtain exact analytic expressions for these coefficients. On the contrary, the coefficients of the terms with the single logarithm and of the terms without logarithm are determined up to a solution of the homogeneous equation, and then they do depend on the two arbitrary constants

$$\tilde{p}_2 \quad \text{and} \quad \tilde{q}_2. \tag{202}$$

The latter can be found by matching the values of the series in eq. (200) with that one in eq. (156) in a point in the range

$$8 < x < 16. \tag{203}$$

By following the matching procedure outlined in the previous section and described in detail in sec. 7.5, we obtain

$$\tilde{p}_2 = -1.04850063265766512303; \tag{204}$$
$$\tilde{q}_2 = +1.50257112894949285675. \tag{205}$$

By decomposing the two-loop integral into different infrared and ultraviolet regions, a large momentum expansion of F_1 has been derived in ref. [15], which reads in our normalization containing an additional factor $1/16$:

$$F_1(x) = \frac{1}{x^2} \left(\frac{7}{192} \log^4 x - \frac{\pi^2}{96} \log^2 x + \frac{5}{4} \zeta(3) \log x - \frac{31}{2880} \pi^4 \right) + O\left(\frac{1}{x^3} \right), \tag{206}$$

where $\zeta(3) \equiv \sum_{n=1}^{\infty} 1/n^3 = 1.20206 \cdots$. By comparing with the first-order terms of our complete solution, we find an analytical agreement in the coefficient of the fourth, triple and double logarithm[8]. As far as the numerically determined coefficients are concerned, differences with the analytical expressions are at most $O(10^{-19})$, well within the expected

---

[8] We thank V. Smirnov for confirming a typo in the coefficient of the $\log^2 x$ term in [15], which must be divided by a factor 3.

numerical uncertainty. We can therefore use in place of our numerical estimates above the values:

$$\tilde{p}_2 = -\frac{31}{2880}\,\pi^4, \qquad \tilde{q}_2 = +\frac{5}{4}\,\zeta(3). \tag{207}$$

Our final expression for the large-momentum expansion of the amplitude $F_1(x)$ of the crossed ladder diagram is that one in eq. (200), with the first coefficients given by:

$$\tilde{p}_2 = -\frac{31}{2880}\,\pi^4\,; \tag{208}$$

$$\tilde{q}_2 = +\frac{5}{4}\,\zeta(3)\,; \tag{209}$$

$$r_2 = -\frac{\pi^2}{96}\,; \tag{210}$$

$$s_2 = 0\,; \tag{211}$$

$$t_2 = \frac{7}{192}\,; \tag{212}$$

$$\tilde{p}_3 = \frac{27}{8} - \frac{5}{48}\,\pi^2 - \frac{31}{1440}\,\pi^4 - \frac{15}{4}\,\zeta(3)\,; \tag{213}$$

$$\tilde{q}_3 = \frac{9}{8} + \frac{\pi^2}{16} + \frac{5}{2}\,\zeta(3)\,; \tag{214}$$

$$r_3 = -\frac{13}{16} - \frac{\pi^2}{48}\,; \tag{215}$$

$$s_3 = -\frac{7}{16}\,; \tag{216}$$

$$t_3 = \frac{7}{96}\,. \tag{217}$$

Let us remark that eqs. (213-217) provide analytic expressions of the coefficients of the leading power-suppressed terms in the large momentum expansion.

7.3.3 Improved Expansion

In order to accelerate the convergence of the series so far obtained, we move from the series in x to the one in the Bernoulli variable

$$t = \log\frac{x-8}{x+1}, \tag{218}$$

with the inverse:

$$\frac{1}{x} = \frac{1-e^t}{8+e^t} = -\frac{1}{9}t - \frac{7}{162}t^2 - \frac{11}{1458}t^3 + \frac{35}{52488}t^4 + \frac{1733}{2361960}t^5 + \frac{7217}{42515280}t^6 + \mathcal{O}(t^7). \tag{219}$$

The convergence radius of the above series is equal to the one of the Bernoulli variable for the expansion around $x = 0$:

$$r_\infty = \sqrt{\pi^2 + 9\log^2 2}. \tag{220}$$

Substituting the r.h.s. of the above equation in the series expansion for $F_1(x)$ obtained previously and finally expanding in powers of t, we have:

$$\tilde{F}_1(t) = \sum_{n=2}^{\infty} \alpha_n t^n + \log x \sum_{n=2}^{\infty} \beta_n t^n + \log^2 x \sum_{n=2}^{\infty} \gamma_n t^n + \log^3 x \sum_{n=2}^{\infty} \delta_n t^n + \log^4 x \sum_{n=2}^{\infty} \phi_n t^n, \quad (221)$$

where the first three coefficients α_n, β_n, γ_n, δ_n, and ϕ_n are:

$$\alpha_2 = \frac{1}{81} a_2; \quad (222)$$

$$\beta_2 = \frac{1}{81} b_2; \quad (223)$$

$$\gamma_2 = \frac{1}{7776} \pi^2; \quad (224)$$

$$\delta_2 = 0; \quad (225)$$

$$\phi_2 = \frac{7}{15552}; \quad (226)$$

$$\alpha_3 = \frac{5}{34992} \pi^2 - \frac{1}{216} + \frac{1}{243} b_2 + \frac{5}{729} a_2; \quad (227)$$

$$\beta_3 = -\frac{1}{11664} \pi^2 - \frac{1}{648} + \frac{5}{729} b_2; \quad (228)$$

$$\gamma_3 = -\frac{5}{69984} \pi^2 + \frac{13}{11664}; \quad (229)$$

$$\delta_3 = \frac{7}{11664}; \quad (230)$$

$$\phi_3 = \frac{35}{139968}; \quad (231)$$

$$\alpha_4 = \frac{95}{1259712} \pi^2 - \frac{2659}{839808} + \frac{5}{2187} b_2 + \frac{49}{26244} a_2; \quad (232)$$

$$\beta_4 = -\frac{5}{104976} \pi^2 - \frac{29}{46656} + \frac{49}{26244} b_2; \quad (233)$$

$$\gamma_4 = -\frac{49}{2519424} \pi^2 + \frac{125}{139968}; \quad (234)$$

$$\delta_4 = \frac{35}{104976}; \quad (235)$$

$$\phi_4 = \frac{343}{5038848}, \quad (236)$$

where (see eq. (199))

$$a_2 = \tilde{p}_2 = -\frac{31}{2880} \pi^4; \qquad b_2 = \tilde{q}_2 = +\frac{5}{4} z(3). \quad (237)$$

The convergence radius of the expansion in t is:

$$\rho_\infty = \log \frac{16}{7} \approx 0.826679. \quad (238)$$

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7.3.4 Analytic Continuation to $x = -\infty$

The expansion of the amplitude $F_1(x)$ for large time-like momenta, namely for (cfr. eq. (172))

$$-\infty < x < -8, \tag{239}$$

can be found from the asymptotic expansion in the space-like region $(x > 8)$ simply by analytic continuation. With the causal prescription

$$x \rightarrow -|x| - i\epsilon, \tag{240}$$

we have in addition to the replacements (103):

$$
\begin{aligned}
\log^3 x &\rightarrow \log^3 |x| - 3\pi^2 \log |x| - 3i\pi \log^2 |x| + i\pi^3 \,; \\
\log^4 x &\rightarrow \log^4 |x| - 6\pi^2 \log^2 |x| + \pi^4 - 4i\pi \log^3 |x| + 4i\pi^3 \log |x| \,.
\end{aligned}
\tag{241}
$$

$F_1(x)$ then becomes complex for $x < 0$ with:

$$
\begin{aligned}
\operatorname{Re} F_1(x) \;=\; & \sum_{n=2}^{\infty} \frac{\tilde{p}_n}{x^n} + \log |x| \sum_{n=2}^{\infty} \frac{\tilde{q}_n}{x^n} + \left(\log^2 |x| - \pi^2\right) \sum_{n=2}^{\infty} \frac{r_n}{x^n} + \tag{242} \\
& + \left(\log^3 |x| - 3\pi^2 \log |x|\right) \sum_{n=2}^{\infty} \frac{s_n}{x^n} + \left(\log^4 |x| - 6\pi^2 \log^2 |x| + \pi^4\right) \sum_{n=2}^{\infty} \frac{t_n}{x^n} \,;
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Im} F_1(x) \;=\; \pi \Bigg[& -\sum_{n=2}^{\infty} \frac{\tilde{q}_n}{x^n} - 2 \log |x| \sum_{n=2}^{\infty} \frac{r_n}{x^n} - \left(3 \log^2 |x| - \pi^2\right) \sum_{n=2}^{\infty} \frac{s_n}{x^n} + \\
& - \left(4 \log^3 |x| - 4\pi^2 \log |x|\right) \sum_{n=2}^{\infty} \frac{t_n}{x^n} \Bigg] \,.
\end{aligned}
\tag{243}
$$

7.4 Expansion Around $x = -1$

In this section we consider the expansion around $x = -1$, the only singularity of the differential equation in the time-like region. Since the nearest singularity to $x = -1$ is the origin $x = 0$, we expect the convergence radius of the expansion around this point to be

$$R_{-1} = 1, \tag{244}$$

i.e. convergence in the circle

$$|x + 1| < 1. \tag{245}$$

On the real axis, that means:

$$-2 < x < 0. \tag{246}$$

7.4.1 Homogeneous Equation

By solving the indicial equation, the solution turns out to be of the form:

$$F_1^{(0)}(x) = \sum_{n=0}^{\infty} a_n (x + 1)^n + \log(x + 1) \sum_{n=0}^{\infty} b_n (x + 1)^n. \tag{247}$$

The differential equation allows one to express all the coefficients of the expansion above in terms of two of them, such as for example the lowest-order ones a_0 and b_0:

$$a_1 = \frac{4}{3}a_0 + \frac{4}{9}b_0 ; \qquad\qquad b_1 = \frac{4}{3}b_0 ; \qquad\qquad (248)$$

$$a_2 = \frac{41}{27}a_0 + \frac{62}{81}b_0 ; \qquad\qquad b_2 = \frac{41}{27}b_0 ; \qquad\qquad (249)$$

$$a_3 = \frac{400}{243}a_0 + \frac{2200}{2187}b_0 ; \qquad\qquad b_3 = \frac{400}{243}b_0 . \qquad\qquad (250)$$

We have a triangular structure of the coefficients, as in the case of the expansion around $x = 0$.

7.4.2 Complete Equation

The expansion of the known term around $x = -1$ is of the form:

$$C(x) = \sum_{n=-1}^{\infty} q_n(x+1)^n + \log(x+1) \sum_{n=0}^{\infty} r_n(x+1)^n . \qquad (251)$$

The leading singularity for $x \to -1$ is therefore a single pole and the subleading singularity is a logarithmic one related to the second series. The first few coefficients read:

$$q_{-1} = \frac{\pi^2}{24} + \frac{\sqrt{5}}{24}K - i\frac{\sqrt{5}}{12}\pi K' ; \qquad (252)$$

$$q_0 = -\frac{1}{4} + \frac{19}{216}\pi^2 + \frac{101}{1080}\sqrt{5}K + i\pi\left(\frac{7}{72} - \frac{101}{540}\sqrt{5}K'\right) ; \qquad (253)$$

$$r_0 = \frac{1}{4} ; \qquad (254)$$

$$q_1 = -\frac{2}{3} + \frac{131}{972}\pi^2 + \frac{1861}{12150}\sqrt{5}K + i\pi\left(\frac{1627}{6480} - \frac{1861}{6075}\sqrt{5}K'\right) ; \qquad (255)$$

$$r_1 = \frac{47}{72} ; \qquad (256)$$

$$q_2 = -\frac{5177}{4320} + \frac{1589}{8748}\pi^2 + \frac{60178}{273375}\sqrt{5}K + i\pi\left(\frac{129527}{291600} - \frac{120356}{273375}\sqrt{5}K'\right) ; \qquad (257)$$

$$r_2 = \frac{749}{648} , \qquad (258)$$

where K and K' are two real transcendental constants defined as:

$$K \equiv \int_0^1 \frac{\log y}{\sqrt{y(y+4)}}dy = -\frac{\pi^2}{5} \approx -1.97392; \qquad (259)$$

$$K' = \log\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.481212. \qquad (260)$$

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Let us note that the coefficients above have an imaginary part related to the 2 massless particles threshold in $s = 0$. Unlike previous cases, the expansion point lies indeed in the time-like region. The particular solution of the inhomogeneous equation is of the same form as the homogeneous one:

$$\bar{F}_1(x) = \sum_{n=0}^{\infty} s_n(x+1)^n + \log(x+1) \sum_{n=0}^{\infty} t_n(x+1)^n . \tag{261}$$

For the coefficients, we find:

$$s_1 = \frac{4}{3}s_0 + \frac{4}{9}t_0 + q_{-1} ; \tag{262}$$

$$t_1 = \frac{4}{3}t_0 ; \tag{263}$$

$$s_2 = \frac{41}{27}s_0 + \frac{62}{81}t_0 - \frac{1}{8} + \frac{59}{864}\pi^2 + \frac{301}{4320}\sqrt{5}K + i\pi\left(\frac{7}{288} - \frac{301}{2160}\sqrt{5}\right) ; \tag{264}$$

$$t_2 = \frac{1}{16} + \frac{41}{27}t_0 ; \tag{265}$$

$$s_3 = \frac{400}{243}s_0 + \frac{2200}{2187}t_0 - \frac{13}{54} + \frac{169}{1944}\pi^2 + \frac{4381}{48600}\sqrt{5}K + i\pi\left(\frac{313}{6480} - \frac{4381}{24300}\sqrt{5}\right) ; \tag{266}$$

$$t_3 = \frac{1}{8} + \frac{400}{243}t_0 ; \tag{267}$$

etc., and we can choose $s_0 = t_0 = 0$. We assume therefore from now on the following expressions of the coefficients:

$$s_0 = 0 ; \tag{268}$$

$$t_0 = 0 ; \tag{269}$$

$$s_1 = \frac{\pi^2}{24} + \frac{\sqrt{5}}{24}K - i\frac{\sqrt{5}}{12}\pi K' ; \tag{270}$$

$$t_1 = 0 ; \tag{271}$$

$$s_2 = -\frac{1}{8} + \frac{59}{864}\pi^2 + \frac{301}{4320}\sqrt{5}K + i\pi\left(\frac{7}{288} - \frac{301}{2160}\sqrt{5}\right) ; \tag{272}$$

$$t_2 = \frac{1}{16} ; \tag{273}$$

$$s_3 = -\frac{13}{54} + \frac{169}{1944}\pi^2 + \frac{4381}{48600}\sqrt{5}K + i\pi\left(\frac{313}{6480} - \frac{4381}{24300}\sqrt{5}\right) ; \tag{274}$$

$$t_3 = \frac{1}{8} . \tag{275}$$

Finally, the general solution is given by:

$$F_1(x) = F_1^{(0)}(x) + \bar{F}_1(x) \tag{276}$$

$$= \sum_{n=0}^{\infty} \tilde{s}_n(x+1)^n + \log(x+1) \sum_{n=0}^{\infty} \tilde{t}_n(x+1)^n , \tag{277}$$

where

$$\tilde{s}_n = a_n + s_n \quad \text{and} \quad \tilde{t}_n = b_n + t_n. \tag{278}$$

Eq. (277) depends on the two arbitrary constants,

$$a_0 \quad \text{and} \quad b_0 ; \tag{279}$$

that can be determined in a numerical way as shown above: we equate the series expansions centered around $x = 0$ and $x = -1$, together with their derivatives, in one point where both series converge, such as for example $x = -1/2$. That way we obtain:

$$a_0 = -0.07572639563476980715 + i\,0.3122156449500221544 ; \tag{280}$$
$$|b_0| < 10^{-18} , \tag{281}$$

naturally implying

$$b_0 = 0. \tag{282}$$

Let us observe that for $x < -1$ the logarithm above acquires an imaginary part related to the 3-particle threshold at $s = m^2$. Then, absorptive contributions related to both thresholds formally come out as imaginary parts of the coefficients of the serieses and, via analytic continuation, of the logarithmic prefactor.

The vanishing of the coefficient b_0 of the $\log(1+x)$ term in the expansion of F_1 (eq. (282)) can be understood as follows. With a proper routing, F_1 can be written as:

$$F_1 = \int \frac{\mathcal{D}^4 k_1 \mathcal{D}^4 k_2}{(q+k_1)^2 + m^2} \frac{1}{(q+k_2)^2 + m^2} \frac{1}{(p_1+k_1)^2} \frac{1}{(p_2+k_2)^2} \frac{1}{(p_1+k_1-k_2)^2} \frac{1}{(p_2-k_1+k_2)^2} . \tag{283}$$

By going to the threshold point $q^2 = -m^2$ and neglecting quadratic terms in the loop momenta $\sim k_1^2,\, k_2^2,\, k_1 \cdot k_2$, one obtains in the soft limit:

$$F_1 \approx \int \mathcal{D}^4 k_1 \mathcal{D}^4 k_2 \, \frac{1}{q \cdot k_1} \frac{1}{q \cdot k_2} \frac{1}{p_1 \cdot k_1} \frac{1}{p_2 \cdot k_2} \frac{1}{p_1 \cdot k_1 - p_1 \cdot k_2} \frac{1}{-p_2 \cdot k_1 + p_2 \cdot k_2} . \tag{284}$$

The integral above is convergent by power counting in the soft region. It is also easy to see that there cannot be any collinear singularity. To have collinear singularities, one should have at least one "all massless" 3-point vertex, i.e. a 3-point vertex connecting massless propagators and/or external lines with light-cone momenta only, which is not the case. Since F_1 has not any infrared singularity in the point $q^2 = -m^2$, it is finite for $x \to -1$[9] and therefore cannot contain any $\log(1 + x)$ term in the expansion, implying $b_0 = 0$.

---

[9] F_1 is ultraviolet finite, as shown previously.

7.4.3 Improved Expansion

In order to improve the convergence of the series so far obtained, we move from the series in x to the one in the Bernoulli variable

$$t \equiv -\log(-x), \tag{285}$$

with inverse:

$$x = -e^{-t} = -1 + t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{120}t^5 - \frac{1}{720}t^6 + \mathcal{O}(t^7). \tag{286}$$

The convergence radius of the series above is, as well known, infinite:

$$r_{-1} = \infty. \tag{287}$$

Substituting the r.h.s. of the above equation in the series expansion for $F_1(x)$ obtained previously and finally expanding in powers of t, we have:

$$\tilde{F}_1(t) \equiv F_1(x(t)) = \sum_{n=0}^{\infty} \alpha_n t^n + \log(x+1) \sum_{n=0}^{\infty} \beta_n t^n, \tag{288}$$

where the first three coefficients α_n and β_n are:

$$\alpha_0 = a_0; \qquad\qquad\qquad\qquad\qquad\qquad \beta_0 = 0; \tag{289}$$

$$\alpha_1 = \frac{4}{3}a_0 + \frac{\pi^2}{24} + \frac{\sqrt{5}}{24}K - i\frac{\sqrt{5}}{12}\pi K'; \qquad\qquad \beta_1 = 0; \tag{290}$$

$$\alpha_2 = -\frac{1}{8} + \frac{41}{864}\pi^2 + \frac{211}{4320}\sqrt{5}K + i\pi\left(\frac{7}{288} - \frac{211}{2160}\sqrt{5}\right); \qquad \beta_2 = \frac{1}{16}. \tag{291}$$

The convergence radius of the expansion in t is given by:

$$\rho_{-1} = \log 2 \approx 0.693147. \tag{292}$$

7.5 Series Expansions Around Regular Points and Additional Matching Points

In the previous sections we illustrated the method of the solution of the differential equation on the MI $F_1(x)$, building series around the singular points only. That is certainly necessary but not sufficient to determine F_1 on the entire real axis. The expansions around all the singular points do not allow indeed to compute the master integral in the range

$$-8 < x < -2. \tag{293}$$

To cover this region, it is necessary to perform expansions of $F_1(x)$ around auxiliary (regular) points x_i, expansions which are not multiplied by any singular function:

$$F_1(x) = \sum_{n=0}^{\infty} c_n^{(i)} (x - x_i)^n. \tag{294}$$

31

It is clear that each new series expansion carries two unknown coefficients, which have to be determined by matching with known serieses, as we have made for the singular points. To cover the region (293), we have constructed two additional series expansions centered in

$$x = -8 \quad \text{and} \quad x = -3. \tag{295}$$

Even though the space-like region $0 < x < 8$ is in principle covered by the expansions around the singular points, we have found it convenient to consider also series expansions centered in

$$x = 1 \quad \text{and} \quad x = 3. \tag{296}$$

The matching procedure is the following[10]:

1. we match the series centered in $x = 1$ with the series centered in $x = 0$, which is completely determined. That way we fix the two arbitrary coefficients entering the expansion in $x = 1$;

2. we then match the series in $x = 3$ with the one in $x = 1$;

3. we finally match the series in $x = 8$ with the series in $x = 3$.

With the above method, we have been able to cover the range $0 < x < 13$ with a numerical precision of about 21 digits. In the range $8 < x < \infty$, we need an additional series expansion in $x = 16$, to be matched with the series centered in $x = 8$.

It is remarkable that the 9 expansion points above (4 singular + 5 regular ones) are sufficient to build a routine evaluating numerically the function $F_1(x)$ for every value of the variable x in the real axis with a relative precision of about 10^{-20}.

The internal consistency of the method can be checked as follows. The series expansion centered in $x = 0$ is completely determined by the initial conditions that we have been able to find. Matching the free parameters of the series expansions centered in $x = -1$ and $x = 1$ with the one centered in $x = 0$ we obtain two other series without free parameters. Now we move along the real positive axis up to $x = \infty$;

$$x = 0 \quad \rightarrow \quad x = 1 \quad \rightarrow \quad x = 3 \quad \rightarrow \quad x = 8 \quad \rightarrow \quad x = \infty. \tag{297}$$

After that, we perform an analytic continuation in order to find the series expansion in $x = -\infty$ and, finally, we reach the series expansion in $x = -1$. What we find is a perfect agreement, within the required precision, between the free parameters that we fixed matching the series expansion in $x = -1$ with the one in $x = 0$ and those found performing the several matches along the real axis, back to $x = -1$ [11]. Another (independent) check of our matching procedure is provided by the values of the coefficients in eqs. (204,205), which are in agreement with the asymptotic expansion of $F_1(x)$ given in [15], as discussed before.

---

[10] The practical advantage of this method with respect to the general analytic continuation by (partially overlapping) circles is related to the use of the differential equation to generate both serieses.

[11] Since we deal in general with multivalued functions, one has to be careful to remain within the same sheet of the Riemann surface by using consistently the causal $+i\epsilon$ prescriptions.

8 Expansions for the Master Integrals F_2 and F_3

The second MI F_2 is directly determined from the first one by means of the (algebraic) equation (33) with $i = 1$, which gives

$$F_2(x) = \frac{1}{2} x \frac{dF_1}{dx}(x) + F_1(x), \tag{298}$$

in which F_1 and F_1' are assumed to be known. One just substitutes the power-series expansions obtained so far for $F_1(x)$ and obtains series representations for $F_2(x)$ in the same convergence domains.

The third MI satisfies a first-order equation of the form

$$\frac{dF_3}{dx}(x) + \frac{1}{x} F_3(x) + N(x) = 0, \tag{299}$$

where the known term can be written as:

$$N(x) = \frac{1}{6x} F_1(x) - \frac{1}{3} \left(1 + \frac{1}{x} \right) F_2(x) + \frac{1}{96x^2} H(-1, 0, 0; x) + \frac{1}{48x^2} H(r, r, 0; x)$$
$$+ \frac{\pi^2}{192x^2} H(-1; x). \tag{300}$$

It involves harmonic polylogarithms as well as F_1 and F_2, which are assumed to be known[12]. The associated homogeneous equation,

$$\frac{dF_3^{(0)}}{dx} + \frac{1}{x} F_3^{(0)}(x) = 0, \tag{301}$$

has the general solution

$$F_3^{(0)}(x) = \frac{K}{x}, \tag{302}$$

where K is an integration constant. A particular solution of the inhomogeneous first-order equation is found in a straightforward manner with the general method of the variation of the constants:

$$\bar{F}_3(x) = -\frac{1}{x} \int_0^x x' N(x') dx'. \tag{303}$$

Since F_1 and F_2 are not given in closed form in terms of known transcendental functions, but only in terms of (truncated) series expansions, one has also to expand all the GHPLs appearing in $N(x)$. As in the case of the first MI F_1, the general solution is:

$$F_3(x) = \frac{K}{x} + \bar{F}_3(x). \tag{304}$$

Analogously to the case of F_1, one can determine analytically all the coefficients of the expansion of F_2 around $x = 0$, by fixing

$$K = 0 \tag{305}$$

---

[12] It is interesting to note that formally F_1 and F_2 appear as sub-topologies in the evaluation of F_3. That also implies that the general method of series expansions allows to compute topologies with subtopologies themselves given by series expansions.

in order to avoid infrared power divergencies. The matching procedure is similar to the one used for F_1; the main difference is that in this case there is only one undetermined coefficient for each series because the differential equation is of first order. Details can be found in [22].

9   Relation to the Equal-Mass Sunrise

There is a conspicuous relation between the homogeneous differential equation for the first master integral F_1 and the homogeneous equation for the equal mass sunrise in two dimensions,

$$S(z) = \int \frac{1}{D_1 D_2 D_3} \mathcal{D}^2 k_1 \mathcal{D}^2 k_2, \tag{306}$$

where:

$$
\begin{aligned}
D_1 &= k_1^2 + m^2, & (307) \\
D_2 &= k_2^2 + m^2, & (308) \\
D_3 &= (p - k_1 - k_2)^2 + m^2, & (309)
\end{aligned}
$$

and

$$z = -\frac{s}{m^2}, \tag{310}$$

with $s = -p^2$ and p^μ the external momentum. The homogeneous equation (see eq. (4.3) of [13]) reads:

$$\frac{d^2}{dz^2} S(z;0) + \left(\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9} \right) \frac{d}{dz} S(z;0) + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)} \right] S(z) = 0. \tag{311}$$

By writing

$$F_1(x) = \frac{1}{x} M(-x-1), \tag{312}$$

whose inverse reads:

$$M(z) = -(z+1) F_1(-z-1), \tag{313}$$

we find that $M(z)$ satisfies the same homogeneous equation of $S(z)$. The "physical origin" of such relation is unclear to us.

10   Conclusions

We have computed the three master integrals for the crossed ladder diagram with two equal-mass quanta exchanged by means of various power-series expansions centered around different points. The two-equal-mass exchanged non-planar topology is by far the most complicated one entering the two-loop form factor and a novel structure emerges, related to the occurrence of elliptic integrals rather than (generalized) harmonic polilogarithms. That was not expected *a priori*, because the diagram (see fig. 1) has only thresholds in $s = 0$ and in $s = m^2$, as well as a pseudothreshold in $s = -4m^2$; we may only generically

relate such structure to the non-planar topology. We argue that the occurrence of elliptic integrals is probably a general feature of massive multi-loop diagrams beyond some level of complexity.

By studying the diagram close to the massless threshold $s = 0$ we have obtained a small momentum expansion with coefficients analitycally determined. By combining our large-momentum expansion with that obtained in ref.[15], we have been able to give the leading power-suppressed corrections to the logarithmic contributions in a completely analytical form.

We have found an *a priori* unexpected relation between the basic (first) master integral for two-mass crossed ladder in four space-time dimensions and the basic master integral for the equal mass sunrise in two dimensions: the homogeneous differential equation is exactly the same, while the inhomogeneous terms are unrelated (in the crossed ladder case, the latter ones being, of course, much more complicated).

This work terminates the computation of the master integrals entering the two-loop electroweak form factor.

Acknowledgments

A   Generalized Harmonic Polylogarithms

The general theory of the harmonic polylogarithms has already been discussed in [11, 10, 23], to which we refer for details. Here we only give the specific information relevant to the computation of the crossed ladder. Let us first give the expressions of the harmonic polylogarithms (HPL's) entering our computation in terms of the standard ones:

$$\begin{align}
H(-1; x) &= \log(1 + x); \tag{314}\\
H(0, 0; x) &= \frac{1}{2} \log^2(x); \tag{315}\\
H(0, -1; x) &= -\operatorname{Li}_2(-x); \tag{316}\\
H(0, -1, 0; x) &= 2\operatorname{Li}_3(-x) - \log x \operatorname{Li}_2(-x). \tag{317}
\end{align}$$

Let us now show that the Generalized Harmonic Polylogarithm (GHPL)[13]

$$H(r, 0; x) \equiv \int_0^x \frac{\log y}{\sqrt{y(4-y)}} \, dy, \tag{320}$$

is real for $0 < x < 4$, purely imaginary for $x > 4$, while it is complex for $x < 0$. In fact, for $0 < x < 4$ the integrand is real in all the integration domain and therefore also H is real. For $x > 4$, since $x \to x - i\epsilon$ with $\epsilon = +0$ and, therefore, $\sqrt{4 - x + i\epsilon} \to i\sqrt{x - 4}$, one can split the integral as:

$$H(r, 0; x) = \int_0^4 \frac{\log y}{\sqrt{y(4-y)}} \, dy - i \int_4^x \frac{\log y}{\sqrt{y(y-4)}} \, dy. \tag{321}$$

The first integral on the r.h.s. vanishes,

$$\int_0^4 \frac{\log y}{\sqrt{y(4-y)}} \, dy = 0, \tag{322}$$

implying that H is purely imaginary in this region. For $x < 0$ the integrand is complex and so is the H. As a consequence, the term entering $C(x)$, the inhomogeneous (and known) term in the differential equation for F_1 given in eq.(54),

$$\frac{H(r, 0; x)}{\sqrt{x(4-x)}} \tag{323}$$

is real for $x > 0$, as it should, while it is complex for $x < 0$.

Let us now show that $H(r, r, 0; x)$ is real for $x > 0$ while it is complex for $x < 0$. Let us write:

$$H(r, r, 0; x) \equiv \int_0^x \frac{1}{\sqrt{y(4-y)}} H(r, 0; y) \, dy. \tag{324}$$

For $0 < x < 4$, the integrand is real in all the integration domain, while for $x > 4$ one can write:

$$H(r, r, 0; x) = \int_0^4 \frac{1}{\sqrt{y(4-y)}} H(r, 0; y) \, dy - i \int_4^x \frac{1}{\sqrt{y(y-4)}} H(r, 0; y) \, dy. \tag{325}$$

Both integrals on the r.h.s. are real.

$H(r, 0; x)$ can be expressed in terms of ordinary harmonic polylogarithms of a non linear function of x, by changing the integration variable from y to

$$t = \frac{y}{2} - 1 + \sqrt{y \left(\frac{y}{4} - 1 \right)} = y - 2 - \frac{1}{y} - \frac{2}{y^2} - \frac{5}{y^3} + O\left(\frac{1}{y^4} \right), \tag{326}$$

---

[13] The weight "r" labels the integration over the function

$$f(r; t) = \frac{1}{\sqrt{t(4-t)}}, \tag{318}$$

such that, for instance:

$$H(r; x) = \int_0^x \frac{dt}{\sqrt{t(4-t)}}. \tag{319}$$

where the square root is the arithmetical one for $y > 4$, and on the last member we have made the expansion for $y \gg 1$. The inverse reads:

$$y = \frac{(t+1)^2}{t}. \tag{327}$$

Note that $y = 4 \leftrightarrow t = 1$. We obtain:

$$H(r,0;x) = -2iH\left[0,-1;\frac{x}{2}-1+\sqrt{x\left(\frac{x}{4}-1\right)}\right] + iH\left[0,0;\frac{x}{2}-1+\sqrt{x\left(\frac{x}{4}-1\right)}\right] + i\frac{\pi^2}{6}. \tag{328}$$

In a similar way, we obtain for the only GHPL of weight 3 the representation:

$$H(r,0,0;x) = -H\left[0,0,0;\frac{x}{2}-1+\sqrt{x\left(\frac{x}{4}-1\right)}\right] + 4H\left[0,-1,0;\frac{x}{2}-1+\sqrt{x\left(\frac{x}{4}-1\right)}\right] +$$
$$+ 6\,\zeta(3), \tag{329}$$

where $H(0,-1,0;1) = -3/2\,\zeta(3)$. In a completely analogous manner one can analyze $H(r,r,0;x)$, the GHPL entering the know term of the differential equation for F_3.

B   Expansion of Generalized Harmonic Polylogarithms

The points $x = 0$, 4 and ∞ are singular points for $H(r,0;x)$. That is easily seen just by looking at the integral definition. The Taylor expansion of $H(r,0;x)$ centered around a *regular* point $x_0 \neq 0$, 4, ∞ is easily obtained by remembering that

$$\frac{d}{dx}H(r,0;x) = f(r,0;x), \tag{330}$$

where

$$f(r,0;x) = \frac{\log x}{\sqrt{x(4-x)}}. \tag{331}$$

By performing the Taylor expansion of $f(r,0;x)$ and integrating on both sides, one obtains:

$$H(r,0;x) = H(r,0;x_0) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f^{(n)}(r,0;x_0)(x-x_0)^{n+1}. \tag{332}$$

Let us note that one transcendental constant only is involved in this expansion, namely $H(r,0;x_0)$. Let us now consider the expansions around singular points. It is clear that $H(r,0;x)$ does not possess an ordinary Taylor expansion around $x = 0$ — the origin is a branch point of infinite order — where derivatives are not defined. This GHPL however has an expansion around zero involving semi-integer powers of x multiplied with up to one power of $\log x$,

$$H(r,0;x) = x^{1/2}\sum_{n=0}^{\infty} k_n\, x^n + x^{1/2}\log x \sum_{n=0}^{\infty} l_n\, x^n, \tag{333}$$

where k_n and l_n are coefficients and the serieses on the r.h.s. have a convergence radius $R = 4$ — i.e. up to the closest singularity. The representation above can be obtained by expanding around $y = 0$ the regular part of the integrand, i.e. the factor $1/\sqrt{4-y}$, so that

$$\frac{\log y}{\sqrt{y(4-y)}} = \frac{1}{2} \log y \, y^{-1/2} + \frac{1}{16} \log y \, y^{1/2} + \frac{3}{256} \log y \, y^{3/2} + \frac{5}{2048} \log y \, y^{5/2} + O\left(y^{7/2}\right)$$

(334)

and using the result:

$$\int \log y \, y^\alpha \, dy = \frac{1}{\alpha+1} \log y \, y^{\alpha+1} - \frac{1}{(\alpha+1)^2} y^{\alpha+1}.$$

(335)

The expansion around $x = 4$ is obtained in an analogous way, i.e. by expanding the regular part of the integrand around $x = 4$, $\log y / \sqrt{y}$, and integrating term by term. An expression of the following form is obtained:

$$H(r, 0; x) = (x-4)^{1/2} \sum_{n=0}^\infty g_n \, (x-4)^n,$$

(336)

where

$$g_0 = 2 \log 2;$$

(337)

$$g_1 = \frac{\log 2}{12};$$

(338)

$$g_2 = \frac{1}{80} - \frac{3}{320} \log 2.$$

(339)

$x = 4$ is therefore a branch point of second order. Let us note therefore that the expression (323) has no singularities for $x \to 4$.

By using the representation obtained in the previous appendix, we can directly obtain the expansion of the GHPL around infinity, which turns out to be of the form:

$$H(r, 0; x) = -i \sum_{n=0}^\infty \frac{a_n}{x^n} - i \log x \sum_{n=0}^\infty \frac{b_n}{x^n} - i \frac{1}{2} \log^2 x,$$

(340)

where the two serieses on the r.h.s. are convergent for $|x| > 4$ and the lowest-order coefficients read:

$$a_0 = \frac{\pi^2}{6}; \qquad\qquad b_0 = 0;$$ (341)

$$a_1 = -2; \qquad\qquad b_1 = -2;$$ (342)

$$a_2 = -\frac{3}{2}; \qquad\qquad b_2 = -3;$$ (343)

$$a_3 = -\frac{20}{9}; \qquad\qquad b_3 = -\frac{20}{3}.$$ (344)

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